

Moderate Deviation Principle for a Class of SPDEs

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Abstract

We establish the moderate deviation principle for the solutions of a class of stochastic partial differential equations with non-Lipschitz continuous coefficients. As an application, we derive the moderate deviation principle for two important population models: super-Brownian motion and Fleming-Viot process.

KEY WORDS: Moderate deviation principle, stochastic partial differential equation, Fleming-Viot process, super-Brownian motion.

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1 Introduction

Many problems in the fields of applications can be modeled by measure-valued processes. Among them are two of the most commonly studied population models called super-Brownian motion (SBM) and Fleming-Viot process (FVP). These population models have been the focus of numerous recent publications. One of the interesting problems on the models is to set the branching rate for SBM and resampling rate for FVP to tend to zero and to study the rate at which the population's measure converges to a deterministic limit. This rate of convergence is best given by the large deviation principle (LDP). In [7], we achieved the LDP for SBM and FVP as the above mentioned rates go to zero and obtained an explicit form of the rate of convergence for each model. However, the topology introduced there is not the natural one. Namely, we used the double quotient space

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due to the non-uniqueness of the controlled PDE in the definition of the rate function. Here we achieve the moderate deviation principle (MDP), which provides the convergence rate of the models as the branching/resampling rate tend to zero at a speed slower than that considered for the LDP. The topology we shall use is the standard one, and there is no need to introduce the quotient space.

MDP for SBM has also been established by Schied in [12]. He used space $\mathcal{C}([0, 1]; M(\mathbb{R}^d))$ equipped with compact open topology, where $M(\mathbb{R}^d)$ is the space of finite signed measures on \mathbb{R}^d with the coarsest topology in which $\mu \mapsto \langle \mu, f \rangle$ are continuous for every bounded Lipschitz function on \mathbb{R}^d . The main tool he applied is the Gärtner-Ellis theorem (cf. Theorem 4.6.1 of [3]). Here we have used a similar space and have obtained the same result; however, with a different approach. Other authors including those of [8, 9, 14, 15] have investigated the MDP for processes related to SBM. These processes include SBM with super-Brownian immigration (SBMSBI) and SBM with immigration governed by Lebesgue measure. Here we have also derived the MDP for FVP, which to our knowledge, has not yet been shown in the literature.

In this article we study the SBM and FVP based on their characterization by solutions to certain SPDEs. We formulate a general class of SPDEs by observing the similarities between the two SPDEs and in Section 3 derive the MDP for this class by applying Theorem 6 of [1]. In Section 4, we then establish the MDP for the two population models with the help of the contraction principle (cf. [4] Theorem 4.2.1). We note that since the formulation of SBM and FVP by SPDEs offered by [13] was given only for dimension one then our result on the MDP is limited to this dimension. For higher dimensions further investigation is required.

2 Notations and Main Results

Suppose (Ω, \mathcal{F}, P) is a probability space and $\{\mathcal{F}_t\}$ is a family of non-decreasing right continuous sub- σ -fields of \mathcal{F} such that \mathcal{F}_0 contains all P -null subsets of Ω . We denote $\mathcal{C}_b(\mathbb{R})$ to be the space of continuous bounded functions on \mathbb{R} and $\mathcal{C}_c(\mathbb{R})$ be the set of continuous functions in \mathbb{R} with compact support. In addition, for $0 < \beta \in \mathbb{R}$, we let $\mathcal{M}_\beta(\mathbb{R})$ denote the set of σ -finite measures μ on \mathbb{R} such that

$$\int e^{-\beta|x|} d\mu(x) < \infty. \quad (1)$$

We endow this space with the topology defined by a modification of the usual weak topology: $\mu^n \rightarrow \mu$ in $\mathcal{M}_\beta(\mathbb{R})$ iff for every $f \in \mathcal{C}_b(\mathbb{R})$,

$$\int_{\mathbb{R}} f(x) e^{-\beta|x|} \mu^n(dx) \rightarrow \int_{\mathbb{R}} f(x) e^{-\beta|x|} \mu(dx).$$

This topology is given by the following modified Wasserstein distance,

$$\rho_\beta(\mu, \nu) := \inf \left\{ \left| \int_{\mathbb{R}} f(x) e^{-\beta|x|} (\mu(dx) - \nu(dx)) \right| : f \in \mathcal{C}_b^1(\mathbb{R}), \|f\|_\infty \vee \|f'\|_\infty \leq 1 \right\}.$$

We denote the probability measures on \mathbb{R} with the above topology by $\mathcal{P}_\beta(\mathbb{R})$. Let (S, \mathcal{S}) be the measurable space defined as

$$(S, \mathcal{S}) := (\mathcal{C}([0, 1]; \mathbb{R}^\infty), \mathbb{B}(\mathcal{C}([0, 1]; \mathbb{R}^\infty))), \quad (2)$$

where \mathbb{R}^∞ is the Polish space with the metric given as

$$d(\{x_i\}, \{y_i\}) := \sum_{i=1}^{\infty} 2^{-i} (|x_i - y_i| \wedge 1).$$

Throughout this paper, we assume $\beta_0 \in (0, \beta)$ and K is a constant which may take different values in different lines. Also notation Δ stands for the second derivative in the spatial variable x . This notation will be used when both spatial and time variables are involved, or when the dual operator will be needed. Otherwise, we will use the simpler notation f'' . Same convention is used for ∇ , the first derivative in spatial variable. For $\alpha \in (0, 1)$, we consider the space $\mathbb{B}_{\alpha, \beta}$ composed of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that for every $m \in \mathbb{N}$, there exist constants $K > 0$ with the following conditions:

$$|f(y_1) - f(y_2)| \leq K e^{\beta m} |y_1 - y_2|^\alpha, \quad \forall |y_1|, |y_2| \leq m \quad (3)$$

$$|f(y)| \leq K e^{\beta|y|}, \quad \forall y \in \mathbb{R} \quad (4)$$

and with the metric,

$$d_{\alpha, \beta}(u, v) = \sum_{m=1}^{\infty} 2^{-m} (\|u - v\|_{m, \alpha, \beta} \wedge 1), \quad u, v \in \mathbb{B}_{\alpha, \beta}$$

where

$$\|u\|_{m, \alpha, \beta} = \sup_{x \in \mathbb{R}} e^{-\beta|x|} |u(x)| + \sup_{y_1 \neq y_2, |y_1|, |y_2| \leq m} \frac{|u(y_1) - u(y_2)|}{|y_1 - y_2|^\alpha} e^{-\beta m}.$$

Note that the collection of continuous functions on \mathbb{R} satisfying (4), referred to as \mathbb{B}_β , is a Banach space with norm,

$$\|f\|_\beta = \sup_{x \in \mathbb{R}} e^{-\beta|x|} |f(x)|.$$

For the convenience of the reader, we now offer a quick introduction to the two population models considered. In SBM model, each individual has an exponentially distributed lifetime and the population evolves as a “cloud.” It is studied by taking a scaled limit of a branching process with an associated branching rate. SBM with branching rate ϵ , denoted by μ_t^ϵ , is a measure-valued Markov process that can be characterized by one of the following.

i) (μ_t^ϵ) having Laplace transform,

$$\mathbb{E}_{\mu_0^\epsilon} \exp(-\langle \mu_t^\epsilon, f \rangle) = \exp(-\langle \mu_0^\epsilon, v(t, \cdot) \rangle),$$

where $v(\cdot, \cdot)$ is the unique mild solution of the evolution equation:

$$\begin{cases} \dot{v}(t, x) = \frac{1}{2} \Delta v(t, x) - v^2(t, x), \\ v(0, x) = f(x), \end{cases}$$

for $f \in \mathcal{C}_p^+(\mathbb{R}^d)$ where for $K > 0$,

$$\mathcal{C}_p(\mathbb{R}^d) := \left\{ f \in \mathcal{C}(\mathbb{R}^d) : |f(x)| < K \phi_p(x) \text{ for } p > d, \phi_p(x) := (1 + |x|^2)^{-\frac{p}{2}} \right\}.$$

ii) (μ_t^ϵ) as the unique solution to a martingale problem given as: for all $f \in \mathcal{C}_b^2(\mathbb{R})$

$$M_t(f) := \langle \mu_t^\epsilon, f \rangle - \langle \mu_0^\epsilon, f \rangle - \int_0^t \left\langle \mu_s^\epsilon, \frac{1}{2} \Delta f \right\rangle ds,$$

is a square-integrable martingale with quadratic variation,

$$\langle M(f) \rangle_t = \epsilon \int_0^t \langle \mu_s^\epsilon, f^2 \rangle ds.$$

iii) In [13] SBM was studied by its “distribution” function-valued process u_t^ϵ defined as

$$u_t^\epsilon(y) = \int_0^y \mu_t^\epsilon(dx), \quad \forall y \in \mathbb{R} \quad (5)$$

and using (5), SBM was characterized by the following stochastic partial differential equation (SPDE),

$$u_t^\epsilon(y) = F(y) + \int_0^t \int_0^{u_s^\epsilon(y)} W(dsda) + \int_0^t \frac{1}{2} \Delta u_s^\epsilon(y) ds, \quad (6)$$

where $F(y) = \int_0^y \mu_0(dx)$ is the “distribution” function of μ_0 , W is an \mathcal{F}_t -adapted space-time white noise random measure on $\mathbb{R}^+ \times \mathbb{R}$ with intensity measure $dsda$.

On the other hand, FVP is a population model with its evolution based on the genetic types of the individuals. It is a probability measure-valued diffusion process studied as a scaled limit of a step-wise mutation model, in which the population size is assumed to stay constant throughout time and individuals move in \mathbb{Z}^d according to a continuous time simple random walk. As in the case for SBM, this population model denoted as (μ_t^ϵ) , is a Markov process and can be characterized by one of the following.

i) (μ_t^ϵ) a family of probability measure-valued Markov process generated by \mathcal{L}^ϵ defined as

$$\begin{aligned} \mathcal{L}^\epsilon F(\mu_t^\epsilon) &= f'(\langle \mu_t^\epsilon, \phi \rangle) \langle \mu_t^\epsilon, A\phi \rangle \\ &\quad + \frac{\epsilon}{2} \int \int f''(\langle \mu_t^\epsilon, \phi \rangle) \phi(x) \phi(y) Q(\mu_t; dx, dy), \end{aligned}$$

for $\epsilon > 0$ where

$$Q(\mu_t^\epsilon; dx, dy) := \mu_t^\epsilon(dx) \delta_x(dy) - \mu_t^\epsilon(dx) \mu_t^\epsilon(dy),$$

with δ_x denoting the Dirac measure at x and A being the generator of a Feller process. The operator \mathcal{L}^ϵ is given on the set,

$$\mathcal{D} = \{F(\mu_t^\epsilon) = f(\langle \mu_t^\epsilon, \phi \rangle) : f \in \mathcal{C}_b^2(\mathbb{R}), \phi \in \mathcal{C}(\mathbb{R})\}.$$

(see [2] and [6] for this formulation).

ii) (μ_t^ϵ) as a unique solution to the following martingale problem: for $f \in \mathcal{C}_c^2(\mathbb{R})$,

$$M_t(f) = \langle \mu_t^\epsilon, f \rangle - \langle \mu_0^\epsilon, f \rangle - \int_0^t \left\langle \mu_s^\epsilon, \frac{1}{2} \Delta f \right\rangle ds,$$

is a continuous square-integrable martingale with quadratic variation,

$$\langle M_t(f) \rangle = \epsilon \int_0^t \left(\langle \mu_s^\epsilon, f^2 \rangle - \langle \mu_s^\epsilon, f \rangle^2 \right) ds.$$

iii) An alternative formulation of FVP was also made in [13]. There by using

$$u_t^\epsilon(y) = \mu_t^\epsilon((-\infty, y]), \quad (7)$$

FVP was proved to be given by the solution to the following SPDE,

$$u_t^\epsilon(y) = F(y) + \int_0^t \int_0^1 (1_{a \leq u_s^\epsilon(y)} - u_s^\epsilon(y)) W(dsda) + \int_0^t \frac{1}{2} \Delta u_s^\epsilon(y) ds. \quad (8)$$

Based on the context, $\epsilon > 0$ represents the branching rate for SBM and resampling rate for FVP. Note that the main difference between (6) and (8) is in the second term; therefore, in [13] a general SPDE with small noise term of the form

$$u_t^\epsilon(y) = F(y) + \sqrt{\epsilon} \int_0^t \int_U G(a, y, u_s^\epsilon(y)) W(dsda) + \int_0^t \frac{1}{2} \Delta u_s^\epsilon(y) ds, \quad (9)$$

was considered with conditions,

$$\int_U |G(a, y, u_1) - G(a, y, u_2)|^2 \lambda(da) \leq K|u_1 - u_2|, \quad (10)$$

$$\int_U |G(a, y, u)|^2 \lambda(da) \leq K(1 + |u|^2), \quad (11)$$

where $u_1, u_2, u, y \in \mathbb{R}$, F is a function on \mathbb{R} and $G : U \times \mathbb{R}^2 \rightarrow \mathbb{R}$. Here we prove the MDP for $\{u_t^\epsilon\}$ by considering the LDP for $\{v_t^\epsilon\}$ given by,

$$v_t^\epsilon(y) := \frac{a(\epsilon)}{\sqrt{\epsilon}} (u_t^\epsilon(y) - u_t^0(y)). \quad (12)$$

Hence we have,

$$v_t^\epsilon(y) = a(\epsilon) \int_0^t \int_U G_s^\epsilon(a, y, v_s^\epsilon(y)) W(dsda) + \frac{1}{2} \int_0^t \Delta v_s^\epsilon(y) ds, \quad (13)$$

where $G_s^\epsilon(a, y, v) := G(a, y, \frac{\sqrt{\epsilon}}{a(\epsilon)}v + u_s^0(y))$ and $a(\epsilon)$ satisfies $0 \leq a(\epsilon) \rightarrow 0$ and $\frac{a(\epsilon)}{\sqrt{\epsilon}} \rightarrow \infty$ as $\epsilon \rightarrow 0$. To form the controlled PDE of (13) we replace the noise by $h \in L^2([0, 1] \times U, ds\lambda(da))$ and obtain

$$v_t(y) = \int_0^t \int_U G(a, y, u_s^0(y)) h(s, a) \lambda(da) ds + \frac{1}{2} \int_0^t \Delta v_s(y) ds. \quad (14)$$

Note that for every $h \in L^2([0, 1] \times U, ds\lambda(da))$, SPDE (14) has a unique solution, which we denote as $\gamma(h)$ for a map $\gamma : L^2([0, 1] \times U, ds\lambda(da)) \rightarrow \mathcal{C}([0, 1]; \mathbb{B}_\beta)$. We are now ready to state the first result of this paper.

Theorem 1. *If $F \in \mathbb{B}_{\alpha, \beta_0}$ for $\alpha \in (0, \frac{1}{2})$ then family $\{v^\epsilon\}$ given by (13) satisfies the LDP in $\mathcal{C}([0, 1]; \mathbb{B}_\beta)$ with rate function,*

$$I(v) = \frac{1}{2} \inf \left\{ \int_0^1 \int_U |h_s(a)|^2 \lambda(da) ds : v = \gamma(h) \right\}, \quad (15)$$

which implies that family $\{u_t^\epsilon\}$ obeys the MDP.

Similar to [5] we consider the Cameron-Martin space which is defined as follows. Let $\mathcal{M}_\beta^S(\mathbb{R})$ be the space of signed measures $\mu = \mu_+ - \mu_-$ with $\mu_\pm \in \mathcal{M}_\beta(\mathbb{R})$. Let \mathcal{D} be the Schwartz space of test functions with compact support in \mathbb{R} and continuous derivatives of all orders. Denote the dual space of real distributions on \mathbb{R} by \mathcal{D}^* then the Cameron-Martin space, H , is composed of $\omega \in \mathcal{C}([0, 1]; \mathcal{M}_\beta^S(\mathbb{R}))$ satisfying the conditions below.

1. $\omega_0 = 0$,
2. the \mathcal{D}^* -valued map $t \mapsto \omega_t$ defined on $[0, 1]$ is absolutely continuous with respect to time. Let $\dot{\omega}$ and $\Delta^* \omega$ be its generalized derivative and Laplacian respectively,
3. for every $t \in [0, 1]$, $\dot{\omega}_t - \frac{1}{2} \Delta^* \omega_t \in \mathcal{D}^*$ is absolutely continuous with respect to μ_t^0 with $\frac{d(\dot{\omega}_t - \frac{1}{2} \Delta^* \omega_t)}{d\mu_t^0}$ being the (generalized) Radon Nikodym derivative,
4. $\frac{d(\dot{\omega}_t - \frac{1}{2} \Delta^* \omega_t)}{d\mu_t^0}$ is in $L^2([0, 1] \times \mathbb{R}, ds\mu(dy))$.

Let \tilde{H} be the space for which conditions for H hold with $\mathcal{M}_\beta^S(\mathbb{R})$ replaced by the space of measures $\mathcal{P}_\beta^S(\mathbb{R})$, and with the additional assumption,

$$\left\langle \mu_t^0, \frac{d(\dot{\omega}_t - \frac{1}{2} \Delta^* \omega_t)}{d\mu_t^0} \right\rangle = 0,$$

where $\mathcal{P}_\beta^S(\mathbb{R})$ is the set of signed measures μ with $\mu_\pm \in \mathcal{P}_\beta(\mathbb{R})$ and $\mu_\pm(\mathbb{R}) =$

1. Denoting $\omega_t^\epsilon(dy) := \frac{a(\epsilon)}{\sqrt{\epsilon}} (\mu_t^\epsilon(dy) - \mu_t^0(dy))$ we have the following two theorems.

Theorem 2. *If $\omega_0 \in \mathcal{M}_\beta(\mathbb{R})$ such that $F \in \mathbb{B}_{\alpha, \beta_0}$ then super-Brownian motion, $\{\mu_t^\epsilon\}$, obeys the MDP in $\mathcal{C}([0, 1]; \mathcal{M}_\beta^S(\mathbb{R}))$ with rate function,*

$$I(\omega) = \begin{cases} \frac{1}{2} \int_0^1 \int_{\mathbb{R}} \left| \frac{d(\dot{\omega}_t - \frac{1}{2} \Delta^* \omega_t)}{d\mu_t^0}(y) \right|^2 \mu_t^0(dy) dt & \text{if } \mu \in H \\ \infty & \text{otherwise.} \end{cases} \quad (16)$$

Theorem 3. Suppose $\omega_0 \in \mathcal{P}_\beta(\mathbb{R})$ such that $F \in \mathbb{B}_{\alpha, \beta_0}$. Then, Fleming-Viot process, $\{\mu^\epsilon\}$, satisfies the MDP on $\mathcal{C}([0, 1]; \mathcal{P}_\beta^S(\mathbb{R}))$ with rate function,

$$I(\omega) = \begin{cases} \frac{1}{2} \int_0^1 \int_{\mathbb{R}} \left| \frac{d(\dot{\omega}_t - \frac{1}{2} \Delta^* \omega_t)}{d\mu_t^0}(y) \right|^2 \mu_t^0(dy) dt & \text{if } \mu \in \tilde{H} \\ \infty & \text{otherwise.} \end{cases} \quad (17)$$

Proofs of Theorems 1-3 are given in Sections 3-5.

3 Moderate Deviations for the General SPDE

Our goal in this section is to establish the MDP for SPDE (9), referred to as the general SPDE. Note that by our assumption $F \in \mathbb{B}_{\alpha, \beta_0}$, we have,

$$|u_s^0(y)| \leq \int_{\mathbb{R}} p_s(x - y) |F(x)| dx \leq K e^{\beta_0 |y|},$$

where $p_t(x) = \frac{1}{\sqrt{2\pi t}} \exp(-\frac{x^2}{2t})$ is the heat kernel. Therefore, G_s^ϵ satisfies conditions,

$$\int_U |G_s^\epsilon(a, y, v_1) - G_s^\epsilon(a, y, v_2)|^2 \lambda(da) \leq K |v_1 - v_2|, \quad (18)$$

$$\int_U |G_s^\epsilon(a, y, v)|^2 \lambda(da) \leq K(1 + v^2 + e^{2\beta_0 |y|}), \quad (19)$$

for $y \in \mathbb{R}$ and $v, v_1, v_2 \in \mathbb{R}$ given by (12).

Since the proof of the uniqueness of strong solutions to SPDE (9) established in [13] only uses condition (10) then the same argument can be applied to SPDE (13) to achieve the uniqueness of strong solutions. SPDE (13) can therefore be presented by its mild form,

$$v_t^\epsilon(y) = a(\epsilon) \int_{\mathbb{R}} \int_0^t \int_U G_s^\epsilon(a, x, v_s^\epsilon(x)) p_{t-s}(y - x) W(ds dx). \quad (20)$$

We show that this mild solution takes values in $\mathcal{C}([0, 1]; \mathbb{B}_\beta)$. To accomplish this we need the subsequent lemma.

Lemma 1. For every $n \geq 2$,

$$\tilde{M} := \sup_{0 < \epsilon < 1} \mathbb{E} \sup_{0 \leq s \leq 1} \left(\int_{\mathbb{R}} |v_s^\epsilon(x)|^2 e^{-2\beta_1 |x|} dx \right)^n < \infty. \quad (21)$$

Proof. We adapt the argument in the proof of Lemma 2.3 of [13] to present setup. By Mitoma [11] if

$$\rho(x) = \begin{cases} C \exp\left(\frac{-1}{1-|x|^2}\right) & |x| < 1 \\ 0 & |x| \geq 1, \end{cases}$$

where C is determined by $\int_{\mathbb{R}} \rho(x) dx = 1$, then $g(x) = \int_{\mathbb{R}} e^{-|y|} \rho(x-y) dy$ satisfies

$$K e^{-|x|} \leq g^{(n)}(x) \leq K e^{-|x|}, \quad (22)$$

where $g^{(n)}(x)$ is the n^{th} derivative of $g(x)$. Note that (22) stays true with $e^{-|x|}$ replaced by $e^{-2\beta_1|x|}$. We then consider $\int J(x) d\mu(x) < \infty$ where $J(x) = \int e^{-2\beta_1|y|} \rho(x-y) dy < \infty$ for the definition of $\mathcal{M}_\beta(\mathbb{R})$ given by (1).

We denote the Hilbert space $L^2(\mathbb{R}, J(x) dx)$ by \mathcal{X}_0 . Applying Itô's formula to (13) we have for every $f \in \mathcal{C}_c^\infty(\mathbb{R}) \cap \mathcal{X}_0$,

$$\begin{aligned} \langle v_t^\epsilon, f \rangle_{\mathcal{X}_0} &= a(\epsilon) \int_{\mathbb{R}} \int_0^t \int_U G_s^\epsilon(a, y, v_s^\epsilon(y)) f(y) J(y) W(ds da) dy \\ &\quad + \int_0^t \langle \frac{1}{2} \Delta v_s^\epsilon, f \rangle_{\mathcal{X}_0} ds, \end{aligned} \quad (23)$$

Itô's formula applied again this time to (23) gives,

$$\begin{aligned} \langle v_t^\epsilon, f \rangle_{\mathcal{X}_0}^2 &= 2a(\epsilon) \int_0^t \langle v_s^\epsilon, f \rangle_{\mathcal{X}_0} \int_U \int_{\mathbb{R}} G_s^\epsilon(a, y, v_s^\epsilon(y)) f(y) J(y) dy W(ds da) \\ &\quad + \int_0^t \langle v_s^\epsilon, f \rangle_{\mathcal{X}_0} \langle \Delta v_s^\epsilon, f \rangle_{\mathcal{X}_0} ds \\ &\quad + a(\epsilon)^2 \int_0^t \int_U \left(\int_{\mathbb{R}} G_s^\epsilon(a, y, v_s^\epsilon(y)) f(y) J(y) dy \right)^2 \lambda(da) ds. \end{aligned} \quad (24)$$

Now we sum over a complete orthonormal system (CONS) of \mathcal{X}_0 , $\{f_j\}_j$ to obtain,

$$\begin{aligned} \|v_t^\epsilon\|_{\mathcal{X}_0}^2 &= 2a(\epsilon) \int_0^t \int_U \langle v_s^\epsilon, G_s^\epsilon(a, \cdot, v_s^\epsilon(\cdot)) \rangle_{\mathcal{X}_0} W(ds da) \\ &\quad + \int_0^t \langle v_s^\epsilon, \Delta v_s^\epsilon \rangle_{\mathcal{X}_0} ds \\ &\quad + a(\epsilon)^2 \int_0^t \int_U \int_{\mathbb{R}} G_s^\epsilon(a, y, v_s^\epsilon(y))^2 J(y) dy \lambda(da) ds. \end{aligned}$$

By Itô's formula,

$$\begin{aligned}
& \|v_t^\epsilon\|_{\mathcal{X}_0}^{2p} \\
&= 2a(\epsilon)p \int_U \int_0^t \|v_s^\epsilon\|_{\mathcal{X}_0}^{2(p-1)} \langle v_s^\epsilon, G_s^\epsilon(a, \cdot, v_s^\epsilon) \rangle_{\mathcal{X}_0} W(dsda) \\
&\quad + \int_0^t p \|v_s^\epsilon\|_{\mathcal{X}_0}^{2(p-1)} \langle v_s^\epsilon, \Delta v_s^\epsilon \rangle_{\mathcal{X}_0} ds \\
&\quad + a(\epsilon)^2 p \int_0^t \|v_s^\epsilon\|_{\mathcal{X}_0}^{2(p-1)} \int_U \int_{\mathbb{R}} G_s^\epsilon(a, y, v_s^\epsilon(y))^2 J(y) dy \lambda(da) ds \\
&\quad + a(\epsilon)p(p-1) \int_0^t \int_U \|v_s^\epsilon\|_{\mathcal{X}_0}^{2(p-2)} \langle v_s^\epsilon, G_s^\epsilon(a, \cdot, v_s^\epsilon(\cdot)) \rangle_{\mathcal{X}_0}^2 \lambda(da) ds.
\end{aligned}$$

Similar to Kurtz and Xiong [10], we can prove that

$$-\int_{\mathbb{R}} v_s^\epsilon(y) (v_s^\epsilon)'(y) J'(y) dy = \frac{1}{2} \int_{\mathbb{R}} v_s^\epsilon(y)^2 J''(y) dy \leq K \|v_s^\epsilon\|_{\mathcal{X}_0}^2$$

and

$$\begin{aligned}
\langle v_s^\epsilon, \Delta v_s^\epsilon \rangle_{\mathcal{X}_0} &\leq -\int_{\mathbb{R}} (v_s^\epsilon)'(y) v_s^\epsilon(y) J'(y) dy \\
&\leq K \|v_s^\epsilon\|_{\mathcal{X}_0}^2.
\end{aligned}$$

Hence with the help of Doob's and Burkholder-Davis-Gundy inequalities we have,

$$\begin{aligned}
& \mathbb{E} \sup_{0 \leq s \leq t} \|v_s^\epsilon\|_{\mathcal{X}_0}^{2p} \\
&\leq Ka(\epsilon) \mathbb{E} \left(\int_0^t \int_U \|v_s^\epsilon\|_{\mathcal{X}_0}^{4(p-1)} \langle v_s^\epsilon, G_s^\epsilon(a, y, v_s^\epsilon(y)) \rangle_{\mathcal{X}_0}^2 ds \lambda(da) \right)^{\frac{1}{2}} \\
&\quad + K \mathbb{E} \int_0^t \|v_s^\epsilon\|_{\mathcal{X}_0}^{2p} ds \\
&\quad + Ka(\epsilon) \mathbb{E} \int_0^t \int_U \int_{\mathbb{R}} \|v_s^\epsilon\|_{\mathcal{X}_0}^{2(p-1)} G_s^\epsilon(a, y, v_s^\epsilon(y))^2 J(y) dy ds \lambda(da) \\
&\quad + Ka(\epsilon) \mathbb{E} \int_0^t \int_U \|v_s^\epsilon\|_{\mathcal{X}_0}^{2(p-2)} \langle v_s^\epsilon, G_s^\epsilon(a, y, v_s^\epsilon(y)) \rangle_{\mathcal{X}_0}^2 ds \lambda(da).
\end{aligned}$$

Now we apply Hölder's inequality and (19) to arrive at

$$\begin{aligned}
\mathbb{E} \sup_{0 \leq s \leq t} \|v_s^\epsilon\|_{\mathcal{X}_0}^{2p} &\leq Ka(\epsilon) \mathbb{E} \left(\int_0^t \|v_s^\epsilon\|_{\mathcal{X}_0}^{4p} ds \right)^{1/2} + K \mathbb{E} \int_0^t \|v_s^\epsilon\|_{\mathcal{X}_0}^{2p} ds \\
&\leq Ka(\epsilon) \mathbb{E} \sup_{0 \leq s \leq t} \|v_s^\epsilon\|_{\mathcal{X}_0}^p \left(\int_0^t \|v_s^\epsilon\|_{\mathcal{X}_0}^{2p} ds \right)^{1/2} \\
&\quad + K \mathbb{E} \int_0^t \|v_s^\epsilon\|_{\mathcal{X}_0}^{2p} ds \\
&\leq \frac{1}{2} \mathbb{E} \sup_{0 \leq s \leq t} \|v_s^\epsilon\|_{\mathcal{X}_0}^{2p} + K_1 \mathbb{E} \int_0^t \|v_s^\epsilon\|_{\mathcal{X}_0}^{2p} ds.
\end{aligned}$$

The conclusion then follows from Gronwall's inequality. \square

For our results, we also apply the lemma given below, the proof of which we have provided in [7].

Lemma 2. *Let $\{X_t^\epsilon(y)\}$ be a family of random fields and suppose $\beta_1 \in (\beta_0, \beta)$. If there exist constants $n, q, K > 0$ such that*

$$\mathbb{E} |X_{t_1}^\epsilon(y_1) - X_{t_2}^\epsilon(y_2)|^n \leq K e^{n\beta_1(|y_1| \vee |y_2|)} (|y_1 - y_2| + |t_1 - t_2|)^{2+q}, \quad (25)$$

then there exists a constant $\alpha > 0$ such that

$$\sup_{\epsilon > 0} \mathbb{E} \left| \sup_m \sup_{t_i \in [0,1], |y_i| \leq m, i=1,2} \frac{|X_{t_1}^\epsilon(y_1) - X_{t_2}^\epsilon(y_2)|}{(|y_1 - y_2| + |t_1 - t_2|)^\alpha} e^{-\beta m} \right|^n < \infty. \quad (26)$$

As a consequence $X^\epsilon \in \mathcal{C}([0,1]; \mathbb{B}_\beta)$ a.s. Furthermore, if condition (25) holds and $\sup_{\epsilon > 0} \mathbb{E} |X_{t_0}^\epsilon(y_0)|^n < \infty$ for some $(t_0, y_0) \in [0,1] \times \mathbb{R}$, then

$$\sup_{\epsilon > 0} \mathbb{E} \left| \sup_{(t,y) \in [0,1] \times \mathbb{R}} e^{-\beta|y|} |X_t^\epsilon(y)| \right|^n < \infty, \quad (27)$$

and the family $\{X^\epsilon\}$ is tight in $\mathcal{C}([0,1]; \mathbb{B}_\beta)$.

Lemma 3. *The solution to SPDE (13) takes values in $\mathcal{C}([0,1]; \mathbb{B}_\beta)$.*

Proof. First we need the following inequalities established in [7]:

$$P_1 := p_{t-s}(y_1 - x) - p_{t-s}(y_2 - x), \quad (28)$$

$$P_2 := p_{t_1-s}(y - x) - p_{t_2-s}(y - x), \quad (29)$$

$$\int_{\mathbb{R}} |P_1|^2 e^{2\beta_1|x|} dx \leq K e^{2\beta_1(|y_1| \vee |y_2|)} (t-s)^{-(\frac{1}{2}+\alpha)} |y_1 - y_2|^\alpha, \quad (30)$$

$$\int_0^{t_1} \int_{\mathbb{R}} |P_2|^2 e^{2\beta_1|x|} dx ds \leq K e^{2\beta_1|y|} |t_1 - t_2|^\alpha, \quad (31)$$

and

$$\int_{t_1}^{t_2} \int_{\mathbb{R}} p_{t_2-s}^2(y-x) e^{2\beta_1|x|} dx ds \leq K |t_1 - t_2|^{\frac{\alpha}{2}} e^{2\beta_1|y|}. \quad (32)$$

We proceed by demonstrating two cases. In case one, we fix $t \in [0, 1]$ and let $y_1, y_2 \in \mathbb{R}$ be arbitrary such that $|y_i| \leq m$ for all $i = 1, 2$. Applying Burkholder-Davis-Gundy and Hölder's inequalities, for $n > 0$ we obtain,

$$\begin{aligned} & \mathbb{E} |v_t^\epsilon(y_1) - v_t^\epsilon(y_2)|^n \\ &= \mathbb{E} \left| a(\epsilon) \int_0^t \int_U \int_{\mathbb{R}} P_1 G_s^\epsilon(a, x, v_s^\epsilon(x)) W(ds da) dx \right|^n \\ &\leq K \mathbb{E} \left(a(\epsilon)^2 \int_0^t \int_U \left(\int_{\mathbb{R}} P_1 G_s^\epsilon(a, x, v_s^\epsilon(x)) dx \right)^2 ds da \right)^{n/2} \\ &\leq K \mathbb{E} \left(a(\epsilon)^2 \int_0^t \int_U \int_{\mathbb{R}} P_1^2 e^{2\beta_1|x|} dx \int_{\mathbb{R}} G_s^\epsilon(a, x, v_s^\epsilon(x))^2 e^{-2\beta_1|x|} dx \lambda(da) ds \right)^{n/2} \\ &\leq K \mathbb{E} \left(a(\epsilon)^2 \int_0^t \int_{\mathbb{R}} P_1^2 e^{2\beta_1|x|} dx \int_{\mathbb{R}} \left(1 + v_s^\epsilon(x)^2 + e^{2\beta_0|x|} \right) e^{-2\beta_1|x|} dx ds \right)^{n/2}. \end{aligned}$$

By (30) we have,

$$\begin{aligned} & \mathbb{E} |v_t^\epsilon(y_1) - v_t^\epsilon(y_2)|^n \quad (33) \\ &\leq K \mathbb{E} \left(\int_0^t e^{2\beta_1(|y_1| \vee |y_2|)} (t-s)^{-(\frac{1}{2}+\alpha)} |y_1 - y_2|^\alpha \int_{\mathbb{R}} v_s^\epsilon(x)^2 e^{-2\beta_1|x|} dx ds \right)^{n/2} \\ &\leq \bar{M} K e^{n\beta_1(|y_1| \vee |y_2|)} |y_1 - y_2|^{\frac{n\alpha}{2}}. \end{aligned}$$

For the second case, we consider $y \in \mathbb{R}$ to be fixed and assume $t_1, t_2 \in [0, 1]$

to be arbitrary, then by (31) and (32),

$$\begin{aligned}
& \mathbb{E} \left| v_{t_1}^\epsilon(y) - v_{t_2}^\epsilon(y) \right|^n \tag{34} \\
& \leq K \mathbb{E} \left| a(\epsilon) \int_0^{t_1} \int_{\mathbb{R}} \int_U P_2 G_s^\epsilon(a, x, v_s(x)) W(dsda) dx \right|^n \\
& \quad + K \mathbb{E} \left| a(\epsilon) \int_{t_1}^{t_2} \int_{\mathbb{R}} \int_U p_{t_2-s}(y-x) G_s^\epsilon(a, x, v_s(x)) W(dsda) dx \right|^n \\
& \leq K \mathbb{E} \left| \int_0^{t_1} \int_{\mathbb{R}} P_2^2 e^{2\beta_1|x|} dx \int_{\mathbb{R}} (K + v_s^\epsilon(x)^2) e^{-2\beta_1|x|} dx ds \right|^{\frac{n}{2}} \\
& \quad + K \mathbb{E} \left| \int_{t_1}^{t_2} \int_{\mathbb{R}} p_{t_2-s}^2(y-x) e^{2\beta_1|x|} dx \int_{\mathbb{R}} (K + v_s^\epsilon(x)^2) e^{-2\beta_1|x|} dx ds \right|^{\frac{n}{2}} \\
& \leq \bar{M} K \left| \int_0^{t_1} \int_{\mathbb{R}} P_2^2 e^{2\beta_1|x|} dx \right|^{\frac{n}{2}} + \bar{M} K \left| \int_{t_1}^{t_2} \int_{\mathbb{R}} p_{t_2-s}^2(y-x) e^{2\beta_1|x|} dx ds \right|^{\frac{n}{2}} \\
& \leq K e^{n\beta_1|y|} |t_1 - t_2|^{\frac{\alpha n}{2}} + K e^{n\beta_1|y|} |t_1 - t_2|^{\frac{\alpha n}{4}} \\
& \leq K e^{n\beta_1|y|} |t_1 - t_2|^{\frac{\alpha n}{4}},
\end{aligned}$$

where in the last step we have used the fact that $|t_1 - t_2| < 1$. \square

We now prove Theorem 1 by applying a technique offered by Budhiraja, et al in [1]. To match their setup, we write SPDE (13) as an infinite sum of independent Brownian motions as follows. Suppose $\{\phi_j\}_j$ is a CONS of $L^2(U, \mathcal{U}, \lambda)$ then,

$$B_t^j := \int_0^t \int_U \phi_j(a) W(dsda), \quad j = 1, 2, \dots \tag{35}$$

is a sequence of independent Brownian motions by Lévy's characterization of Brownian motions. We can then present SPDE (13) in the following form,

$$v_t^\epsilon(y) = a(\epsilon) \sum_j \int_0^t G_s^{\epsilon,j}(y, v_s^\epsilon(y)) dB_s^j + \frac{1}{2} \int_0^t \Delta v_s^\epsilon(y) ds, \tag{36}$$

where

$$G_s^{\epsilon,j}(y, v) := \int_U G_s^\epsilon(a, y, v) \phi_j(a) \lambda(da). \tag{37}$$

Similarly, the controlled PDE (14) can be written as

$$v_t(y) = \sum_j \int_0^t \int_U G(a, y, u_s^0(y)) k_s^j \phi_j(a) \lambda(da) ds + \frac{1}{2} \int_0^t \Delta v_s(y) ds, \tag{38}$$

where

$$k_s^j := \int_U h_s(a) \phi_j(a) \lambda(da).$$

By the same argument as in [13], SPDE (36) has a strong solution so there exists a map $g^\epsilon : \mathbb{B}_{\alpha, \beta_0} \times S \rightarrow \mathcal{C}([0, 1]; \mathbb{B}_\beta)$ such that $v^\epsilon = g^\epsilon(a(\epsilon)B)$ where $B = \{B_t^j\}$. We now define

$$\mathcal{S}^N(\ell_2) := \{k \in L^2([0, 1], \ell_2) : \int_0^1 \|k_s\|_{\ell_2}^2 ds \leq N\}. \quad (39)$$

To verify the assumption imposed by Theorem 6 in [1] let $\{k^\epsilon\}$ be a family of random variables taking values in $\mathcal{S}^N(\ell_2)$ such that $k^\epsilon \rightarrow k$ in distribution as $\epsilon \rightarrow 0$ and consider the SPDE,

$$\begin{aligned} v_t^{\theta, \epsilon}(y) &= \theta \sum_j \int_0^t \int_{\mathbb{R}} p_{t-s}(y-x) G_s^{\epsilon, j}(x, v_s^{\theta, \epsilon}(x)) dB_s^j dx \\ &\quad + \sum_j \int_0^t \int_{\mathbb{R}} p_{t-s}(y-x) G_s^{\epsilon, j}(x, v_s^{\theta, \epsilon}(x)) k_s^{\epsilon, j} dx ds. \end{aligned} \quad (40)$$

We establish the tightness of $\{v^{\theta, \epsilon}\}$ as follows.

Lemma 4. $v_t^{\theta, \epsilon}(y)$ is tight in $\mathcal{C}([0, 1], \mathbb{B}_\beta)$.

Proof. According to Lemma 2, to achieve the tightness for $\{v_t^{\theta, \epsilon}\}$, it is sufficient to show estimate (25) for $v_t^{\theta, \epsilon}$ and verify that $\sup_{\epsilon > 0} \mathbb{E} \left| v_{t_0}^{\theta, \epsilon}(y_0) \right|^n < \infty$ for some $(t_0, y_0) \in [0, 1] \times \mathbb{R}$. Following the same steps as in the proof of lemma 1, we have

$$\tilde{M} := \sup_{0 < \epsilon < 1} \mathbb{E} \sup_{0 \leq s \leq 1} \left(\int_{\mathbb{R}} \left| v_s^{\theta, \epsilon}(x) \right|^2 e^{-2\beta_1|x|} dx \right)^n < \infty. \quad (41)$$

Notice that estimate (25) can be attained for the first term on the right hand side of (40) by exactly the same calculations done in Lemma 3 with the use of \tilde{M} given in (41) instead of \bar{M} of Lemma 1. Thus, we focus on finding estimate (25) for

$$\tilde{v}_t^{\theta, \epsilon}(y) := \sum_j \int_0^t \int_{\mathbb{R}} p_{t-s}(y-x) G_j^{\epsilon, j}(x, v_s^{\theta, \epsilon}(x)) k_s^{\epsilon, j} dx ds.$$

Using the same method used in the proof of Lemma 3, we begin by fixing $t \in [0, 1]$ and assuming y_1, y_2 to be any real numbers such that $|y_i| \leq m$ for $i = 1, 2$ and $m \in \mathbb{N}$. Recall

$$P_1 := p_{t-s}(y_1 - x) - p_{t-s}(y_2 - x).$$

With the help of Cauchy-Schwartz inequality, (19), and our result (30), we obtain the estimates below,

$$\begin{aligned}
& \mathbb{E} \left| \tilde{v}_t^{\theta, \epsilon}(y_1) - \tilde{v}_t^{\theta, \epsilon}(y_2) \right|^n \\
&= \mathbb{E} \left| \int_0^t \int_{\mathbb{R}} P_1 \sum_j G_s^{\epsilon, j}(x, v_s^{\theta, \epsilon}(x)) k_s^{\epsilon, j} dx ds \right|^n \\
&\leq \mathbb{E} \left| \int_0^t \int_{\mathbb{R}} P_1 \left(\sum_j G_s^{\epsilon, j}(x, v_s^{\theta, \epsilon}(x))^2 \right)^{\frac{1}{2}} \|k_s^{\epsilon}\|_{\ell_2} dx ds \right|^n \\
&\leq \mathbb{E} \left| \left(\int_0^t \left(\int_{\mathbb{R}} P_1 \sqrt{K \left(1 + v_s^{\theta, \epsilon}(x)^2 + e^{2\beta_0|x|} \right)} dx \right)^2 ds \right)^{\frac{1}{2}} \left(\int_0^t \|k_s^{\epsilon}\|_{\ell_2}^2 ds \right)^{\frac{1}{2}} \right|^n \\
&\leq \mathbb{E} \left| \int_0^t \left(\int_{\mathbb{R}} P_1 \sqrt{K \left(1 + v_s^{\theta, \epsilon}(x)^2 + e^{2\beta_0|x|} \right)} dx \right)^2 dx \right|^{\frac{n}{2}} N^{\frac{n}{2}} \\
&\leq K e^{n\beta_1(|y_1| \vee |y_2|)} |y_1 - y_2|^{\frac{\alpha n}{2}},
\end{aligned}$$

where $N > 0$ is the constant given by (39). Furthermore, the case for $0 \leq t_1 < t_2 \leq 1$ arbitrary and $y \in \mathbb{R}$ fixed can be given by

$$\begin{aligned}
\mathbb{E} \left| \tilde{v}_{t_1}^{\theta, \epsilon}(y) - \tilde{v}_{t_2}^{\theta, \epsilon}(y) \right|^n &\leq K \mathbb{E} \left| \int_{t_1}^{t_2} \int_{\mathbb{R}} P_2 \sum_j G_s^{\epsilon, j}(x, v_s(x)) k_s^{\epsilon, j} dx ds \right|^n \\
&\quad + K \mathbb{E} \left| \int_0^{t_1} \int_{\mathbb{R}} p_{t_2-s}^2(y-x) \sum_j G_s^{\epsilon, j}(x, v_s(x)) k_s^{\epsilon, j} dx ds \right|^n \\
&\leq K e^{n\beta_1|y|} |t_1 - t_2|^{\frac{n\alpha}{4}},
\end{aligned}$$

where,

$$P_2 := p_{t_1-s}(y-x) - p_{t_2-s}(y-x).$$

□

Thus, $\{v_t^{\theta, \epsilon}\}$ is tight and for the assumption of Theorem 6 of [1] to be satisfied we let $\theta = 0$ for its first part and $\theta = a(\epsilon)$ for the second part and apply the Prohorov Theorem and so by their Theorem 6, our Theorem 1 can be deduced.

4 Moderate Deviations for SBM and FVP

We devote this section to the proofs of Theorems 2 and 3. Recall

$$\omega_t^\epsilon(dy) := \frac{a(\epsilon)}{\sqrt{\epsilon}} (\mu_t^\epsilon(dy) - \mu_t^0(dy)), \quad (42)$$

where in the case of SBM, $u_t^\epsilon(y) := \int_0^y \mu_t^\epsilon(dx)$. Then based on (12), we can write $v_t^\epsilon(y) := \int_0^y \omega_t^\epsilon(dx)$. Similarly, for FVP we have $u_t^\epsilon(y) := \int_{-\infty}^y \mu_t^\epsilon(dx)$ which gives $v_t^\epsilon(y) := \int_{-\infty}^y \omega_t^\epsilon(dx)$. Analogous to Lemma 6 of [7] we have that for the set of functions with finite variations, \mathcal{A} , the map $\xi : \mathbb{B}_\beta \cap \mathcal{A} \rightarrow \mathcal{M}_\beta^S(\mathbb{R})$ given as $\xi(u)(B) = \int 1_B(y) du(y)$ for all $B \in \mathbb{B}(\mathbb{R})$ is continuous. Therefore, map $\eta : \mathcal{C}([0, 1]; \mathbb{B}_\beta) \rightarrow \mathcal{C}([0, 1]; \mathcal{M}_\beta^S(\mathbb{R}))$ defined as $\eta(v)_t = \xi(v_t)$ is also continuous. Since SBM and FVP can be written as $\omega_t^\epsilon(dy) = \eta(v^\epsilon)_t([0, y])$ and $\omega_t^\epsilon(dy) = \eta(v^\epsilon)_t((-\infty, y])$ respectively, then in both cases $\omega_t^\epsilon(y)$ is a continuous function of v_t^ϵ . Based on our LDP result for v_t^ϵ given in Theorem 1, we can conclude by the contraction principle that $\{\omega_t^\epsilon\}$ also satisfies the LDP for both models.

Our remaining task is to identify an explicit representation of the models' MDP rate functions. According to the contraction principle, rate functions for SBM and FVP are given by $\inf \{I(u) : u \in \eta^{-1}(\omega)\}$. Since η is injective, we then aim to find the rate functions following the form given by (15).

As for SBM, formulation (6) satisfies the general SPDE (9) with the following properties,

$$U = \mathbb{R}, \quad \lambda(da) = da, \quad G(a, y, u) = 1_{0 \leq a \leq u} + 1_{u \leq a \leq 0},$$

then using the controlled PDE (14) we have,

$$\begin{aligned}
& \langle \omega_t, f \rangle = \langle \partial_x v_t, f \rangle = - \langle v_t, f' \rangle \\
& = - \int_0^t \int_0^\infty \int_0^{u_s^0(y)} h_s(a) f'(y) da dy ds - \int_0^t \int_{-\infty}^0 \int_{u_s^0(y)}^0 h_s(a) f'(y) da dy ds \\
& \quad - \int_0^t \langle \frac{1}{2} \Delta v_s, f' \rangle ds \\
& = \int_0^t \int_0^\infty h_s(a) f((u_s^0)^{-1}(a)) da ds - \int_0^t \int_{-\infty}^0 h_s(a) f((u_s^0)^{-1}(a)) da ds \\
& \quad + \int_0^t \langle \frac{1}{2} \omega_s, \Delta f \rangle ds \\
& = \int_0^t \int_0^\infty h_s(u_s^0(y)) f(y) du_s^0(y) ds - \int_0^t \int_{-\infty}^0 h_s(u_s^0(y)) f(y) du_s^0(y) ds \\
& \quad + \int_0^t \langle \frac{1}{2} \Delta^* \omega_s, f \rangle ds \\
& = \int_0^t \langle h_s(u_s^0) \text{sgn}(\cdot) \mu_s^0, f \rangle ds + \frac{1}{2} \int_0^t \langle \Delta^* \omega_s, f \rangle ds.
\end{aligned}$$

Thus,

$$h_t(u_t^0(y)) \text{sgn}(y) = \frac{d(\dot{\omega}_t - \frac{1}{2} \Delta^* \omega_t)}{d\mu_t^0}(y).$$

Notice that

$$\int_{\mathbb{R}} |h_t(a)|^2 da = \int_{\mathbb{R}} |h_t(u_t^0(y))|^2 du_t^0(y) = \int_{\mathbb{R}} |h_t(u_t^0(y))|^2 d\mu_t^0(y).$$

Letting the right hand side of (16) be denoted as $I_0(\mu)$, if $I(\mu) < \infty$ then $I(\mu)$ given in (15) with $U = \mathbb{R}$ is equal to $I_0(\mu)$. For the case $I_0(\mu) < \infty$ we can reverse the above calculations to obtain $I_0(\mu) = I(\mu)$.

Similarly for FVP, since FVP satisfies the general SPDE (9) with

$$U = [0, 1], \quad \lambda(da) = da, \quad G(a, y, u) = 1_{a < u} - u,$$

then

$$\begin{aligned}
& \langle \omega_t, f \rangle = - \langle v_t, f' \rangle \\
&= - \int_0^t \int_{\mathbb{R}} \int_0^{u_s^0(y)} h_s(a) f'(y) da dy ds \\
&\quad + \int_0^t \int_{\mathbb{R}} \int_0^1 u_s^0(y) h_s(a) f'(y) da dy ds - \int_0^t \langle \frac{1}{2} \Delta v_s(y), f' \rangle ds \\
&= \int_0^t \langle h_s(u_s^0) \mu_s^0, f \rangle ds - \int_0^t \langle \int_0^1 h_s(a) da \mu_s^0, f \rangle ds \\
&\quad + \int_0^t \langle \frac{1}{2} \Delta^* \omega_s, f \rangle ds.
\end{aligned}$$

Thus,

$$\dot{\omega}_t - \frac{1}{2} \Delta^* \omega_t = h_t(u_t^0(y)) \mu_t^0 - \int_0^1 h_t(a) da \mu_t^0.$$

Our goal is to find the infimum of $\int_0^1 |h_s(a)|^2 da$ over $h_s(a)$ satisfying (14). We note that if h satisfies (14) then $g_s(a) := h_s(a) - \int_0^1 h_s(a) da$ also satisfies the same equation. It is well-known that the second moment is minimized when it is centralized. Therefore, we replace $h_s(a)$ by $g_s(a)$ in the definition of the rate function, and obtain it as,

$$\int_0^1 |g_s(a)|^2 da = \int_0^1 \left| \frac{d(\dot{\omega}_t - \frac{1}{2} \Delta^* \omega_t)}{d\mu_t^0}(y) \right|^2 d\mu_t^0(y),$$

in (15) to arrive at (17) for the case $I(v) < \infty$ and based on a similar argument as in the case of SBM we obtain (17). Thus, MDP is proved for the two models.

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